S é minaire Lotharingien de Combinatoire **5 1** (2004) , Article B 5 1 f

EULER - MAHONIAN PARAMETERS ON COLORED PERMUTATION GROUPS

Eli Bagno
Einstein Institute of Mathematics , The Hebrew University
Givat Ram , Israel
E - mail : bagnoe @ math . huj i . ac . i l

Abstract . New combinatorial statistics on colored permutation groups are introduced here . We present two different generaliza - tions of major index and descent number , one of them is com - binatorial in nature and the other is algebraic . We also present Euler - Mahonian type j oint distributions of our parameters .

1 . Introduction

One of the most active branches in Enumerative Combinatorics i s the study of permutation statistics . Let S_n be the symmetric group on n letters and let $fi : S_n \to \boldsymbol{Z}_+, (1 \leq i \leq t)$ be (non - negative , integer valued) combinatorial parameters . Then one i s interested in the refined enumeration of permutations according to these parameters :

$$\sum_{\pi \in S_n} q_1^{f1(\pi)} \cdots q_t^{ft(\pi)},$$

where qi are variables . Some of the most important parameters are the following : A *descent* in a permutation $\pi = \pi_1, ..., \pi_n$ i s a position i such that $\pi(i+1) < \pi(i)$, an *inversion* is a pair $i < j$ such that $\pi(i) > \pi(j)$, and the *major index* of π is the sum of it s descents . The last parameter was introduced by MacMahon in [9] and [1 0] . He called it the " greater index " . He proved algebraically that the major index and the inversion number are equi - distributed over the symmetric group . In other words :

$$\sum_{\pi \in S_n} q^{inv(\pi)} = \sum_{\pi \in S_n} q^{maj(\pi)} = [n]_q!,$$

where $[n]_q = \frac{1-q^n}{1-q}$. (In fact , MacMahon proved the same result for the more general case where S_n is replaced by the set of all rearrangements

of a given word - see [1 1] . The first combinatorial proof of this result was given by Foata [5]) .

A permutation statistic that i s equi - distributed with the number of descents is called *Eulerian* , while a permutation statistic that is equi - distributed with the inversion number is called *Mahonian.*

A natural extension of the study of permutation statistics is the study of pairs of permutation statistics and their j oint distributions . Of particular interest is the j oint distribution of the descent number and the major index . The generating function for this j oint distribution is given by Carlitz ' s q - Eulerian polynomial :

$$A_n(t,q) = \sum_{\pi \in S_n} t^{des(\pi)} q^{maj(\pi)} = \prod_{i=0}^{n} (1-tq^i) \sum_{k \geq 0} [k+1]_q^n t^k.$$

(See[4], [7]).

The following problem was first suggested by Foata :

Problem 1 . 1 . *(Foata) , Extend the Euler - Mahonian distribution of de - s cent number and major index t o the hyperoctahedral group* B_n.

A so lut ion of this problem was given by Adin , Brenti and Roichman [1] meanwhile generalizing the concept of major index in two different ways , one of them algebraic in nature ($fmaj$) and the other combina - torial in nature ($nmaj$). Let $G_{r,n} = \mathbb{Z}_r \wr S_n$ be the group of colored permutations (see Section 2 . 2 below) . In this paper we further extend the major index to $G_{r,n}$ in two different ways :

• The parameter $lmaj$ i s equi - distributed with the length function of $G_{r,n}$. We prove the following (See Theorem 5 . 2) :

Theorem .

$$\sum_{\pi \in G_{r,n}} q^{lmaj(\pi)} = \sum_{\pi \in G_{r,n}} q^{\ell(\pi)}$$

where ℓ *is the length function with respect to the s tandard generators of* $G_{r,n}$ *(s ee Section 2 . 2 below) .*

We define also the parameter $ldes$ which i s a length - oriented gener - alization of the descent number . The parameters $ldes$ and $lmaj$ have Euler - Mahonian type j o int distribution . We prove (see Theorem 5 . 3) :

Theorem .

$$\frac{\sum_{\pi \in G_{r,n}} q^{lmaj(\pi)} t^{ldes(\pi)}}{n \prod_{i=0}^{n} (1-tq^i) \Pi_{k=1}^{n} (1+q^k t [r-1]_{qt})} = \sum_{k \geq 0} [k+1]_q^n t^k.$$

• The second direction is to generalize the negative descent number , $ndes$, and the negative major index $,nmaj$, defined for B_n in [1] . The pair $(ndes, nmaj)$ has a s impler Euler - Mahonian type j o int distribu - t ion . We prove (s ee Theorem 6 . 8) :

Theorem .

$$\sum_{\in_\pi G_{r,n}} q^{nmaj(\pi)} t^{ndes(\pi)} = \frac{(t^r;q^r)_{n+1}}{[r]t} \sum_{k\geq 0} [k+1]_q^n t^k$$

The notation $(t^r;q^r)_n + 1$ *will be explained in Section 2 . 1 .*

The rest of this paper is organized as follows : In the next section we present some needed notations to be used in the s equel , including the colored permutation groups . In Section 3 we present some basic statis - t ics on them . In Section 4 we present a formula for the length function of $G_{r,n}$ (see also [1 3]) . In Section 5 we introduce the parameters $ldes$ and $lmaj$ and find their j oint distributions . The parameters $nmaj$ and $ndes$ will be presented in Section 6 along with their j o int distribution . It should be noted that , unlike $lmaj, nmaj$ is not equi - distributed with the standard length function .

2 . Preliminaries

2 . 1 . **Notations .** Let $\mathbb{N} := \{0,1,2,3,...\}$ and let $\mathbb{Z}$ be the r ing of integers ,$\mathbb{Q}$ be the field of rational numbers and $\mathbb{C}$ the field of complex numbers .

For $a \in \mathbb{N}$ let $[a] := \{1,2,...,a\}$ (where $[0] := \varnothing$). The cardinality
of a set A will be denoted by $|A|$. More generally , given a multiset

$$r$$

$M = \{1^a 1, 2^a 2, ..., r^{a_r}\}$, denote by $|M|$ it s cardinality , so $|M| = \sum a_i$.

$$i = 1$$

Given a variable q and a commutative r ing R, denote by $R[q]$ (re - spectively , $R[[q]]$) the r ing of polynomials (respectively , formal power series) in q with coefficients in R.

Define :

$$(a;q)_n := \begin{cases} 1 \quad \text{if} n = 0; \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) \quad \text{otherwise.} \end{cases}$$

Also , let :

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$$

(so $[0]_q = 0$)and

$$[n]_q! := \frac{(q;q)_n}{(1-q)^n} = [n]_q \cdot [n-1]_q \cdots [1]_q.$$

Let n be a non - negative integer . A *partition* of n i s an infinite
sequence of non - negative integers with finitely many non - zero t erms

$$\lambda = (\lambda_1, \lambda_2, ...), \text{where} \lambda_1 \geq \lambda_2 \geq \cdots \qquad \text{and} \sum_{i=1}^{\infty} \lambda_i = n.$$

The sum $\sum \lambda_i = n$ i s called the *size* of λ, denoted $\mid \lambda \mid$; write also $\lambda \vdash n$. The number of parts of $\lambda, \ell(\lambda)$, is the maximal j for which $\lambda_j > 0$. The unique partition of $n = 0$ is the *empty partition* $\varnothing = (0, 0, ...)$, which has length $\ell(\varnothing) := 0$.

For any partition λ with at most n positive parts let

$$m_j(\lambda) :=\mid \quad \{1 \leq i \leq n \quad \mid \quad \lambda_i = j\} \quad \mid \quad (\text{forall} j \geq 0),$$

and let $\binom{n}{\bar{m}(\lambda)}$ denote the multinomial co efficient $\binom{n}{m_0(\lambda), m1(\lambda), ...}$.

2 . 2 . **The Group of Colored Permutations .**

Definition 2 . 1 . Let r and n be positive integers . The group of colored permutations of n digits with r colors i s the wreath product $G_{r,n} =$
$\mathbb{Z}_r \wr S_n = \mathbb{Z}_r^n \rtimes S_n$, consisting of all the pairs (z, τ) where z is an $n-$ tuple of integers between 0 and $r-1$ and $\tau \in S_n$. The multiplication i s defined by the fo llowing rule : For $z = (z_1, ..., z_n)$ and $z' = (z'_1, ..., z'_n)$

$$(z, \tau) \cdot (z', \tau') = ((z_1 + -^{\tau}_{z'} 1_{(1)}, ..., z_n + -^{\tau}_{z'} 1_{(n)}), \tau \circ \tau')$$

(here $+$ is taken mod r).

Here are some conventions we use along this paper :

For an element $\pi = (z, \tau) \in G_{r,n}$ with $z = (z_1, ..., z_n)$ we write $z_i(\pi) = z_i$. For $\pi = (z, \tau)$, we write $\mid \pi \mid = (0, \tau), (0 \in \mathbb{Z}_r^n)$.

A much more natural way to present $G_{r,n}$ is the fo llowing : Consider the alphabet $\Sigma = \{1, ..., n, \bar{1}, ..., \bar{n}, ..., 1^{[r-1]}, ..., n^{[r-1]}\}$ as the s et $[n]$ colored by the colors $0, ..., r-1$. Then an element of $G_{r,n}$ is a *co lored permutation* , i . e . , a bij ection $\pi : \Sigma \to \Sigma$ such that $\pi(\bar{i}) = -\pi_{(i)}$. In this manner we write , for example , the colored permutation $(z, \tau) = ((1, 0, 3, 2), (2, 1, 4, 3)) \in G_{3,4}$ as $\begin{pmatrix} 1 & 2 & 3 & 4 \\ \bar{2} & 1 & \bar{\bar{4}} & \bar{\bar{3}} \end{pmatrix}$ or even j ust as : $\bar{2}1\bar{\bar{4}}_{\bar{\bar{3}}}$.

The group $G_{r,n}$ i s generated by the set $S_{G_{r,n}} = \{s_0, s_1, ..., s_{n-1}\}$
defined by their action on the set $[n]$ as follows :

$$s_i(j) := \begin{cases} i+1, & \text{if} \quad j = i; \\ i, & \text{if} \quad j = i+1; \\ j, & \text{otherwise.} \end{cases}$$

 whereas the exceptional generator s_0 i s defined by

$$s_0(j) := \begin{cases} \bar{1}, & \text{if } j = 1; \\ j, & \text{otherwise.} \end{cases}$$

Note that , unlike the case of Coxeter groups , here the set of genera - tors i s not symmetric . Note also that $G_{1,n} = C_1 \wr S_n$ i s the symmetric group S_n while $G_{2,n} = C_2 \wr S_n$ i s the group of s igned permutations , also known as the hyperoctahedral group , or the classical Weyl group of
type B .

Returning now to the view of $G_{r,n}$ as a semidirect product : $\mathbb{Z}_r^n \rtimes S_n$, we note that it would be worthwhile to consider $\mathbb{Z}_r^n$ in another way : Given any order on the alphabet Σ, $\mathbb{Z}_r^n$ can be identified with the subgroup T of $G_{r,n}$ consisting of the ordered permutations , i . e . those satisfying : $i < j \Longrightarrow \pi(i) < \pi(j)$ with respect to the given order . (This is done by sending a vector $z = (z_1, ..., z_n) \in \mathbb{Z}_r^n$ to the unique ordered permutation $\pi \in G_{r,n}$ satisfying $z_i(\pi) = z_i$. For example , given the
order $\bar{3} < \bar{2} < \bar{1} < 1 < 2 < 3$ on the set $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$, the vector
$(0, 1, 1) \in \mathbb{Z}_2^3$ corresponds to the colored permutation $(^{\bar{3}\,\bar{2}}1) \in G_{2,3}$). Note that i f we denote by $\mathbb{S}$ the group , i somorphic to S_n, generated by $S = S_{G_{r,n}} - \{s_0\}$, then T is j ust a s et of coset representatives of $\mathbb{S}$.
(Indeed , any colored permutation $\pi \in G_{r,n}$ can be written uniquely in the following way : $\pi = \sigma \cdot u$ where σ i s an ordered permutation and

$$u \in \mathbb{S}).$$

3 . Basic Statistics on $G_{r,n}$ For $\pi \in G_{r,n}$ we define the *negative s et* of π by :

$$Neg(\pi) = \{i \mid z_i(\pi) \neq 0\}. \tag{1}$$

The s iz e of the set $Neg(\pi)$ will be denoted by $neg(\pi)$. The fo llowing parameters we define on $G_{r,n}$ depend on the assumption that we have some order on the alphabet Σ.
Let $\pi \in G_{r,n}$. We say that the pair $i < j$ i s an inversion of π if $\pi(i) > \pi(j)$. The number of inversions in π i s denoted by $inv\ (\pi)$.

$i \in [n-1]$ is a *des cent* of π i f $\pi(i) > \pi(i+1)$. We define :

$$Des(\pi) := \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$$

to be the *des cent s et* of π and we denote by $des(\pi)$ the s ize of $Des(\pi)$. We also let

$$maj(\pi) := \sum_{i \in Des(\pi)} i$$

and call it the *major index* of π.

For example , consider the order $\bar{1} < \bar{2} < \bar{3} < 1 < 2 < 3$ defined on the set $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$. Also consider the colored permutation :

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ \bar{3} & 2 & \bar{1} \end{pmatrix}.$$

Then one has $inv(\pi) = 2, des(\pi) = 1$ and $maj(\pi) = 2$.

Now , given any order on the alphabet Σ, we write $\pi = \sigma \cdot u$ where $\sigma \in T$ and $u \in \mathbb{S}$(T and $\mathbb{S}$ were defined at the end of Section 2 . 2) . We present a few s imple facts concerning this coset decomposition which will be used later .

$$des(\pi) = des(u). \tag{2}$$

$$neg(\pi) = neg(\sigma). \tag{3}$$

$$\sum_{i=1}^{n} z_i(\pi) = \sum_{i=1}^{n} z_i(\sigma). \tag{4}$$

$$maj(\pi) = maj(u). \tag{5}$$

$$\sum_{z_i(\pi) \neq 0} | \pi(i) | = \sum_{z_i(\sigma) \neq 0} | \sigma(i) | . \tag{6}$$

4 . A Length function for $G_{r,n}$

In this section we present a formula for the length of $G_{r,n}$ with respect to $S_{G_{r,n}}$. A s imilar expression appears in [1 3] . The proof of our formula depends on the length order we define next although the length function it s elf i s independent of order . We start with the definition of a length function for $G_{r,n}$:

Definition 4 . 1 . For every $\pi \in G_{r,n}$ define the *length* of π with respect to the set of the generators $S_{G_{r,n}} = \{s_0, s_1, ..., s_{n-1}\}$ to be the minimal number of generators satisfying that their product i s π. Formally :

$$\ell(\pi) = \min\{r \in N : \pi = s_{i_1} \cdots s_{i_r}, \text{forsome} i_1, ..., i_r \in [0, n-1]\}.$$

Definition 4 . 2 . The *length order* on the alphabet

$$\Sigma = \{1, ..., n, \bar{1}, ..., \bar{n}, ..., 1^{[r-1]}, ..., n^{[r-1]}\}$$

is defined as follows :

$$n^{[r-1]} < \cdots < \bar{n} < \cdots < 1^{[r-1]} < \cdots < \bar{1} < 1 < \cdots < n.$$

Theorem 4 . 3 . *For every* $\pi \in G_{r,n}$:

$$\ell(\pi) = inv(\pi) + \sum_{z_i(\pi) \neq 0} (| \pi(i) | + z_i(\pi) - 1)$$

where $inv(\pi)$ *is calculated with respect to the length order defined above .*

 Proof . Denote

$$L(\pi) = inv(\pi) + \sum_{z_i(\pi) \neq 0} (\mid \pi(i) \mid + z_i(\pi) - 1).$$

We prove first that $\ell(\pi) \leq L(\pi)$ by presenting an algorithm which expresses π as a product of $L(\pi)$ generators . For $\pi \in G_{r,n}$ write $\pi = \sigma u$ where $\sigma \in T, u \in \mathbb{S}$. Our algorithm sends the identity permutation first to σ and then from σ to $\sigma u = \pi$. This i s done by multiplying on the r ight by Coxeter generators .

Start with the identity permutation .

- For every $j = \mid \pi(i) \mid$ such that $i \in Neg(\pi)$, in increasing order

of$j$:

$-$ Move j to place 1 by multiplying on the r ight by the $j-1$ successive decreasing generators : $s_{j-1}, ..., s_1$. $-$ Equip j with $z_i(\pi)$ bars , to get $\pi(i)$. This will be done by

multiplying on the r ight by $s_0^{z_i(\pi)}$ and thus will cost exactly

$z_i(\pi)$steps.

After doing this process once for every ‘ colored digit ‘ we get $\sigma \in T$, i . e .,σ is increasing according to the length order . • Mix the permutation σ in order to get π out of it . This will

cost$inv(\pi)$steps.

Example : $\pi = \bar{\bar{3}}^{\bar{2}} 4 \bar{1}$. Here $\sigma = \bar{\bar{3}}^{\bar{2}\bar{1}} 4$, and $u = 1243$. The process is (Multiplication i s always on the r ight) :

$$1234 s_0 \rightarrow \bar{1}234 s_1 \rightarrow 2\bar{1}34 s_0 \rightarrow \bar{2}\bar{1}34 s_2 \rightarrow \bar{2}3^{\bar{1}} 4 s_1 \rightarrow 3^{\bar{2}\bar{1}} 4 s_0 \rightarrow \bar{3}^{\bar{2}\bar{1}} 4 s_0 \rightarrow \bar{\bar{3}}^{\bar{2}\bar{1}} 4 = \sigma.$$

Now , we are left with an increasing ordered permutation which we have to mix . This will be done by $inv(\pi)$ elements of the generating set $S = S_{G_{r,n}} - \{s_0\}$ which form u and thus we have $\pi = \sigma u$. In

n

summary , we used $inv(\pi) + \sum (\mid \pi(i) \mid -1) + \sum z_i$ steps . This proves

$$z_i(\pi) \neq 0 \quad i = 1$$
$$\ell(\pi) \leq L(\pi).$$

We prove now the other direction . For $r = 2, \mathbb{Z}_r \wr S_n$ i s the Coxeter group of type B. In this case it i s known that $L(\pi) = \ell(\pi)$ (See for

example[3]).

Take $r > 2$ and let $\pi = ((z_1, ..., z_n), \tau) \in G_{r,n}$. We construct $\pi' =$

$$((z'_{1,},...,z'_n),\tau) \in G_{2,n} \text{bydefining :}$$

$$z'_i = \begin{cases} 0 & z_i = 0; \\ 1 & z_i > 0. \end{cases}$$

For example , if $\pi = \bar{\bar{4}}_{\bar{3}2\bar{1}} \in G_{3,4}$ then $\pi' = \bar{4}_{\bar{3}2\bar{1}} \in G_{2,4}$.

$n \quad n$

Denoting $k = \sum z_i - \sum z'_i$ we have , by the construction of π', that

$i = 1 \quad i = 1$

$L(\pi) = L(\pi') + k$. Now , assume to the contrary that $L(\pi) > \ell(\pi)$. We have $\ell(\pi) \geq \ell(\pi') + k$. (Indeed , take any reduced word representing π and delete from it the k occurrences of the generator s_0, which are responsible for coloring the digits in π that are reducing colors in the passage to π'. This will give us a word representing π').

We have now :

$$L(\pi') + k = L(\pi) > \ell(\pi) \geq \ell(\pi') + k$$

which contradicts the fact that $L(\pi') = \ell(\pi')$ in $G_{2,n}$. □ We proceed to the calculation of the generating function of the length function .

Theorem 4 . 4 .

$$\sum_{\pi \in G_{r,n}} q^{\ell(\pi)} = [n]_q! \cdot \prod_{i=1}^{n} (1 + q^i [r-1]_q).$$

Proof . If $\pi \in G_{r,n}$ then using the coset decomposition we can write $\pi = \sigma \cdot u$ where $\sigma \in T, u \in \mathbb{S}$. As can be easily deduced by the proof of Theorem 4 . 3 , the length function can also be written as :

$$\ell(\pi) = inv(u) + \ell(\sigma)$$

where $\ell(\sigma) = \sum_{z_i(\sigma) \neq 0} (\mid \sigma(i) \mid -1 + z_i(\sigma))$ i s the length function of σ.

Now , it i s well known that $\sum_{u \in \mathbb{S}} q^{inv(u)} = [n]_q!$. Thus ,

$$\sum_{\pi \in G_{r,n}} q^{\ell(\pi)} = \sum_{u \in \mathbb{S}} q^{inv(u)} \cdot \sum_{\sigma \in T} q^{\ell(\pi)} = [n]_q! A_n$$

where by induction

$$A_n = (1 + q^{n-1+1} + q^{n-1+2} + \cdots + q^{n-1+r-1}) A_{n-1}.$$

We have in summary :

$$\sum_{\pi \in G_{r,n}} q^{\ell(\pi)} = [n]_q! \prod_{i=1}^{n} (1 + q^i [r-1]_q).$$

□

5 . The Parameter *lmaj*

In this section we introduce the first generalization of the parameter *maj* defined for the symmetric groups and prove it s equi - distribution with the length function of $G_{r,n}$. We introduce also the parameter *ldes* which i s a generalization of the parameter *des* defined for the symmetric groups . We start with the following definitions :

Definition 5 . 1 . For every $\pi \in G_{r,n}$ we define :

$$ldes(\pi) = des(\pi) + \sum_{i=1}^{n} z_i(\pi)$$

and

$$lmaj(\pi) = maj(\pi) + \sum_{z_i(\pi)\neq 0 \quad i=1}^{n} (\mid \pi(i) \mid -1) + \sum z_i(\pi)$$

where the descents are computed with respect to the length order .

Theorem 5 . 2 . *The parameter lmaj equi distributes with the length function over* $G_{r,n}$, *i . e . ,*

$$\sum_{\pi \in G_{r,n}} q^{\ell(\pi)} = \sum_{\pi \in G_{r,n}} q^{lmaj(\pi)}.$$

Proof . We use the semidirect decomposition of $G_{r,n}$ and the equations (3) , (4) , (5) and (6) to get :

$$\begin{aligned}
\sum_{\pi \in G_{r,n}} q^{lmaj(\pi)} &= \sum_{\sigma u \in G_{r,n}} q^{maj(\sigma u)+\sum_{z_i(\sigma u)\neq 0}^{P}(\mid \sigma u(i) \mid -1 + z_i(\sigma u))} \\
&= \sum q^{maj(u)} \sum q^{z_i P_{(\sigma)\neq 0}(\mid \sigma(i) \mid -1 + z_i(\sigma))} \\
&\qquad u \in \mathbb{S} \quad \sigma \in T \\
&= \sum q^{inv(u)} \sum q^{z_i P_{(\sigma)\neq 0}(\mid \sigma(i) \mid -1 + z_i(\sigma))} \\
&\qquad u \in \mathbb{S} \quad \sigma \in T \\
&= \sum_{\pi \in G_{r,n}} q^{\ell(\pi)}.
\end{aligned}$$

□

5 . 1 . **Euler - Mahonian Type Distribution .** In this section we pre - sent an Euler - Mahonian type bi - distribution for the parameters *lmaj* and *ldes* over $G_{r,n}$.

Theorem 5 . 3 .

$$\frac{\sum_{\pi \in G_{r,n}} t^{ldes(\pi)} q^{lmaj(\pi)}}{(t;q)_n + 1(1+q^k t[r-1]_{qt})} = \sum_{k \geq 0} [k+1]_q^n t^k.$$

Proof . We use the semidirect decomposition of $G_{r,n}$ and the equations (2) , (3) , (4) , (5) and (6) to get :

$$\begin{aligned}
\sum_{\pi \in G_{r,n}} t^{ldes(\pi)} q^{lmaj(\pi)} &= \sum_{\pi \in G_{r,n}} t^{des(\pi) + P_n^{i=1} z_i(\pi)} q^{maj(\pi) + {}_{z_i} P_{(\pi) \neq} 0(|\ \pi(i)\ | -1 + z_i(\pi))} \\
&= \sum_{\sigma \in Tu \in \mathbb{S}} \sum t^{des(u) + P_n^{i=1} z_i(\sigma)} q^{maj(u) + {}_{z_i} P_{(\sigma) \neq} 0(|\ \sigma(i)\ | -1 + z_i(\sigma))} \\
&= \sum P_n^{ti=1} z_i(\sigma)^P_{q^{z_i(\sigma)}) \neq 0(|\ \sigma(i)\ | -1 + z_i(\sigma))} \sum_{\sigma \in T \quad u \in \mathbb{S}} t^{des(u)} q^{maj(u)}.
\end{aligned}$$

By [1 , Theorem 2 . 2] , we have :

$$\sum_{u \in \mathbb{S}} t^{des(u)} q^{maj(u)} = \prod_{i=0}^{n} (1 - tq^i) \sum_{k \geq 0} [k+1]_q^n t^k, \tag{7}$$

so we are left with the sum :

$$\sum_{\sigma \in T} t P_n^{i=1} z_i(\sigma)^P_{q^{z_i(\sigma)}) \neq 0(|\ \sigma(i)\ | -1 + z_i(\sigma)).}$$

By the same t echnique we adopted in the proof of Theorem 4 . 4 we can prove that

$$\sum_{\sigma \in T} P_n^{ti=1} z_i(\sigma)^P_{q^{z_i(\sigma)}) \neq 0(|\ \sigma(i)\ | -1 + z_i(\sigma)) = n \prod_{i=1} (1 + q^i t[r-1]_{qt}).}$$

Combining this with equation (7) we get :

$$\sum_{\pi \in G_{r,n}} t^{ldes(\pi)} q^{lmaj(\pi)} = \prod_{k=1}^{n} (1 + q^k t[r-1]_{qt}) \prod_{i=0}^{n} (1 - tq^i) \sum_{k \geq 0} [k+1]_q^n t^k,$$

and thus

$$\frac{\sum_{\pi \in \mathbb{Z}_r \wr S_n} t^{ldes(\pi)} q^{lmaj(\pi)}}{\Pi_{i=0}^{n}(1-tq^i)\Pi_{k=1}^{n}(1+q^k t[r-1]_{qt})} = \sum_{k\geq 0}[k+1]_q^n t^k.$$

□

6 . The Parameter $nmaj$

In this section we define another parameter on $\mathbb{Z}_r \wr S_n$. This parameter
is a generalization of the parameter $nmaj$ defined in [2] . The results of this section do not depend depend on the order one chooses on the

$$\text{alphabet}\Sigma = \{1, ..., n, \bar{1}, ..., \bar{n}, ..., 1^{[r-1]}, ..., n^{[r-1]}\}.$$

Definition 6 . 1 . For $\pi \in \mathbb{Z}_r \wr S_n$ define the multiset :

$$NNeg(\pi) = \{i^{z_i(\pi)}\}.$$

Note that each i with $z_i(\pi) > 0$ appears $z_i(\pi)$ t imes . **Definition 6 . 2 .** For $\pi \in G_{r,n}$ define the multiset :

$$NDes(\pi) = Des(\pi) \uplus NNeg(\pi^{-1}).$$

We define also $ndes(\pi) =\mid NDes(\pi) \mid$. **Definition 6 . 3 .** For $\pi \in G_{r,n}$ define

$$nmaj(\pi) = \sum_{i \in NDes(\pi)} i,$$

and also

$$ndes(\pi) =\mid NDes(\pi) \mid .$$

Example 6 . 4 . *Consider the order* $\bar{1} < \bar{2} < \bar{3} < 1 < 2 < 3$ *and take*

$\pi = \bar{3}^{\bar{1}}2$. $HereDes(\pi) = \{1\}$, $\pi^{-1} = \bar{2}3^{\bar{1}}$, $NNeg(\pi^{-1}) = \{1, 3, 3\}$,
$NDes(\pi) = \{1, 1, 3, 3\}, nmaj(\pi) = 1 + 1 + 3 + 3 = 8 and ndes(\pi) = 4.$

We define also some refinements of the parameters $ndes$ and $nmaj$. They will be used in the proof of the main result of this section . **Definition 6 . 5 .** For every $\pi \in G_{r,n}$ define

$$d_i(\pi) :=\mid \{j \in Des(\pi) : j \geq i\} \mid \quad (1 \leq i \leq n).$$

This i s the number of descents in π from position i on . Define also for every $\pi \in G_{r,n}$

$$n_i(\pi) =\mid \{j \in NNeg(\pi) : j \geq i\} \mid .$$

Note that $n_i(\pi)$ counts the number of colors from position i on . **Observation 6 . 6 .** *Let* $\pi \in G_{r,n}$. *Then*

$$ndes(\pi) = d_1(\pi) + n_1(\pi^{-1}),$$

$$nmaj(\pi) = \sum_{i=1}^{n} [d_i(\pi) + n_i(\pi^{-1})].$$

6 . 1 . **Euler - Mahonian Distribution .** In what follows we present an Euler - Mahonian type distribution of the parameters $ndes$ and $nmaj$ de - fined earlier . We use here the Hilbert series of the algebra $\mathbb{C}[q1, ..., qn]$ with respect to multi - degree rearranged into a weakly decreasing se - quence , i . e . , a partition . The r ight - hand s ide in the following result i s the Hilbert series of the above algebra .

Theorem 6 . 7 .

$$\sum_{(\ell\lambda)\leq n} \left(\begin{array}{c} n \\ \bar{m}(\lambda) \end{array} \right) \prod_{n}^{i=1} q_i^{\lambda_i} = \frac{\sum_{\pi \in G_{r,}\, n} \prod_{i=1} q_i^{d_i(\pi)+n_i(\pi^{-1})}}{n \prod (1 - q_1^r \cdots q_i^r)}$$

$$i = 1$$

$$in \mathbb{C}[[q1, ..., qn]].$$

Proof . Recall from Section 3 the definition of $T \subseteq G_{r,n}$ as the set of ordered permutations . As already shown in Section $3, T$ can be s een as
a copy of $\mathbb{Z}_r^n$ and thus the fact that $G_{r,n} = \mathbb{Z}_r^n \rtimes S_n$ implies that every $\pi \in G_{r,n}$ can be uniquely written as $\pi = \sigma u$ where $\sigma \in T$ and $u \in \mathbb{S}$, where $\mathbb{S}$ is the subgroup of $G_{r,n}$ generated by $S_{G_{r,n}} - \{s_0\}$.

It is clear from the definitions that $d_i(\sigma u) = d_i(u)$ and $n_i(u^{-1}\sigma^{-1}) = n_i(\sigma^{-1})$ for all $\sigma \in T, u \in S_n$ and $1 \leq i \leq n$. Therefore

$$\begin{aligned} \sum \prod q_i^{d_i(\pi)+n_i(\pi^{-1})} &= \sum \sum \prod q_i^{d_i(\sigma u)+n_i((\sigma u)^{-1})} \\ \pi \in G_{r,n} i = 1 \quad u \in \mathbb{S} \sigma \in T i = 1 \\ &= \sum \sum \prod q_i^{d_i(u)+n_i(\sigma^{-1})} \\ u \in \mathbb{S} \sigma \in T i = 1 \\ &= \sum \prod q_i^{d_i(u)} \cdot \sum \prod q_i^{n_i(\sigma^{-1})}. \\ u \in \mathbb{S} i = 1 \quad \sigma \in T i = 1 \end{aligned}$$

In [2 , Theorem 6 . 2] , it is proven that the Hilbert series with respect to multi - degree of $\mathbb{C}[q1, ..., qn]$ can be written as a product of the gen - erating function of the descent basis for type A and the generating function of the symmetric functions . Explicitly :

$$\sum_{(\ell\lambda)\leq n} \left(\begin{array}{c} n \\ \bar{m}(\lambda) \end{array} \right) \prod_{n}^{i=1} q_i^{\lambda_i} = \frac{\sum_{\pi \in S_n} \prod_{i=1} q_i^{d_i(\pi)}}{n \prod (1 - q1 \cdots qi)} \tag{8}$$

$$i = 1$$

in $\mathbb{C}[[q1, ..., qn]]$, where the sum on the left - hand s ide is taken over all partitions with at most n parts .

Thus , in order to complete the proof we have to prove that :

$$\sum_{\sigma \in T} \prod_{n}^{i=1} q_i^{n_i(\sigma^{-1})} = \frac{\prod_{i=1}(1 - q_1^r \cdots q_i^r)}{n \prod (1 - q1 \cdots qi)} \quad (9)$$

$$i = 1$$

Define a function $\phi : \{0, ..., r-1\} \to \{0, .., r-1\}$ by

$$\phi(i) = \begin{cases} 0 & i = 0 \\ r - i & i \neq 0 \end{cases} .$$

Note that there i s a bij ection between the elements $\sigma \in T$ and the multisets of the form $\{1^j 1, ..., n^{j_n}\}$ where $0 \leq ji \leq r-1$, given by

$$\sigma \mapsto NNeg(\sigma^{-1}) = \{| \, \sigma(i) \, |^{\phi(z_i)(\sigma)}\}.$$

(Indeed , given any multiset $A = \{1^j 1, ..., n^{j_n}\}$, define τ by $\tau(i) = i_{\phi(j_i)}$ for every $1 \leq i \leq n$ and then order τ to get σ. For example , given $A = \{1, 1, 2, 2, 3\}$, we form $\tau = \bar{1}\bar{2}\bar{3}4$ and then $\sigma = \bar{\bar{3}}\bar{\bar{2}}\bar{\bar{1}}4$).

Now , in order to calculate the sum $\sum n \prod q_i^{n_i(\sigma^{-1})}$, we can run over

$$\sigma \in Ti = 1$$

the multisets of the form $\{1^j 1, ..., n^{j_n}\}$ where $0 \leq ji \leq r-1$. Here , every i, inserted to such a multiset ji t imes , contributes the monomial $(q1 \cdots qi)^{ji}$ to our sum . This gives us :

$$\begin{aligned} & n \\ & \sum \prod q_i^{n_i(\sigma^{-1})} \\ & \sigma \in Ti = 1 \\ & = (1 + q1 + \cdots + q_1^{r-1})(1 + q1q2 + (q1q2)^2 + \cdots + (q1q2)^{r-1}) \\ & \cdots (1 + q1 \cdots qn + (q1 \cdots qn)^2 + \cdots + (q1 \cdots qn)^{r-1}) \\ & n \\ & \prod (1 - q_1^r \cdots q_i^r) \\ & i = 1 \\ & \hline \\ & = \quad n \quad . \\ & \prod (1 - q1 \cdots qi) \\ & i = 1 \\ & \square \end{aligned}$$

This leads us to the following generalization of the Carlitz identity

for the parameters *ndes* and *nmaj*.

Theorem 6 . 8 .

$$\sum_{\in_\pi G_{r,}} {}_n q^{nmaj(\pi)} t^{ndes(\pi)} = \frac{(t^r;q^r)_n + 1}{[r]t} \sum_{k\geq 0} [k+1]_q^n t^k.$$

 Proof . Substitute , in Theorem 6.7, $q1 = qt$ and $q2 = q3 = \cdots = qn = q$
to get :

$$\sum_{(\ell\lambda)\leq n} \left(\begin{array}{c} n \\ \bar{m}(\lambda) \end{array} \right) P_n^{q^{i=1}} \lambda_i \cdot t^{\lambda_1} = \frac{\sum_{\in \pi G_{r,}\, n} q^{nmaj(\pi)} t^{ndes(\pi)}}{n \prod (1 - tr_q ir)} .$$

$i = 1$

Dividing by $(1 - t)$ we have :

$$\frac{\sum_{\in \pi G_{r,n}} q^{nmaj(\pi)} t^{ndes(\pi)}}{(1-t)\prod_n^{i=1}(1 - tr_q ir)} = \sum_{\infty}^{k=0} \sum_{\substack{(\lambda) \\ \lambda_\ell 1 \leq \leq_k^n}} \left(\begin{array}{c} n \\ \bar{m}(\lambda) \end{array} \right) P_n^{q^{i=1}} \lambda_{i_{t^k,}}$$

and thus the coefficient of t^k i s

$$\sum_{(\ell\lambda)\leq n} \left(\begin{array}{c} n \\ \bar{m}(\lambda) \end{array} \right) P_n^{q^{i=1}} \lambda_i = (l_1, ... \sum_{,l_n)\in} [0,k]^n P_n^{q^{i=1}} l i = (\sum_k^{j=0} q^j)^n = [k+1]_q^n,$$

$\lambda_1 \leq k$

and we have proved :

$$\sum_{\pi \in G_{r,n}} q^{nmaj(\pi)} t^{ndes(\pi)} = (1-t) \prod_{i=1}^{n} (1 - t^r q^{ir}) \sum_{k \geq 0} [k+1]_q^n t^k$$
$$= \frac{(t^r; q^r)_n + 1}{[r]t} \sum_{k \geq 0} [k+1]_q^n t^k.$$

□

Appendix A . The distribution of *nmaj*

After the completion of this work we were told about the preprint of J . Haglund , N . Loehr and J . B . Remmel [8] : ' Statistics on wreath prod - ucts , perfect matchings and s igned words ' . A variant of the parameter defined here as *nmaj* appears also in their work . Its distribution over

$$G_{r,n}$$

i s proven there to be $\prod_{i=1}^{n} [ri]_q$. We enclose here another proof of this distribution , based on the proof of Theorem 6 . 8 of the last section .

Theorem Appendix A . 1 .

$$\sum_{\pi \in G_{r,n}} q^{nmaj(\pi)} = \prod_{i=1}^{n} [ri]_q.$$

Proof . We write :

$$\sum_{\pi \in G_{r,n}} q^{nmaj(\pi)} = \sum_{\pi \in G_{r,n}} Pi_{q=1}^{\sum_{i=1}^{n} d_i(\pi) + n_i(\pi^{-1})}$$

$$= \sum_{u \in \mathbb{S}} Pi_{q=1}^{\sum_{i=1}^{n} d_i(u)} \sum_{\sigma \in T} Pi_{q=1}^{\sum_{i=1}^{n} n_i(\sigma^{-1})}.$$

By substituting $q = q1 = \cdots = qn$ in equation 8 we get :

$$\sum_{u \in \mathbb{S}} P_n^{\sum_{i=1}^{n} d} i(u) = Hilb(\mathbb{C}[x_1, ..., x_n], q) \cdot n \prod_{i=1} (1 - q^i)$$

but on the other hand

$$Hilb(\mathbb{C}[x_1, ..., x_n], q) = \frac{1}{(1-q)^n}$$

so we have

$$\sum_{u \in \mathbb{S}} P_n^{\sum_{i=1}^{n} d} i(u) = \prod_{n}^{i=1} (1 - q^i) \cdot \frac{1}{(1-q)^n}.$$

We turn now to the calculation of $\sum_{\sigma \in T} Pi_{q=1}^{\sum_{i=1}^{n} n_i(\sigma^{-1})}$.

Substituting $q = q1 = \cdots = qn$ in equation 9 we get :

$$\sum_{\sigma \in T} P_n^{\sum_{i=1}^{n} n} i(\sigma^{-1}) = \frac{\prod_{i=1}(1 - q^{ri})}{n \prod (1 - q^i)}.$$

We have now :

$$\sum_{\in_\pi G_{r,}} {}_n q^{nmaj(\pi)} = \prod_{n}^{i=1} (1 - q^i) \cdot \frac{1}{(1-q)^n} \cdot \frac{\prod_{i=1}(1 - q^{ri})}{n \prod (1 - q^i)} = \prod_{n}^{i=1} [ri]_q.$$

□

Acknowledgments . I want to thank Ron Adin and Yuval Roichman - for useful discussions and fruitful ideas .

References

[1] R . M . Adin , F . Brenti and Y . Roichman , *Descent numbers and major indices for the hyperoctahedral group* , Adv . Appl . Math . , 27 (2001) , 210 – 224 .

[2] R . M . Adin , F . Brenti and Y . Roichman , *Descent representations and multivariate Statistics* , Trans . Amer . Math . Soc . , to appear .

[3] A . Bj ö rner and F . Brenti , *Combinatorics of Coxeter groups* , preprint .

[4] L . Carlitz , *A combinatorial property of q - Eulerian numbers* , Amer . Math . Monthly , 82 (1 975) , 5 1 – 54 .

[5] D . Foata , *Distributions Euleriennes e t Mahoniennes sur le groupe des per - mutations* , M . Aigner (ed) , Higher Combinatorics , 27 – 49 , Reidel , Boston , Berlin Combinatorics Symposium , (1 976) .

[6] D . Foata , *Rearrangements of words* , in M . Lothaire , Combinatorics on words , Encyclopedia of Math . and its Appl . , Vol . 1 7 , Addison - Wesley Pub - lishing Company , (1 983) .

[7] I . M . Gessel , *Generating functions and enumeration of s equences* , Ph . D . Thesis , M . I . T , 1 977 .

[8] J . Haglund , N . Loehr and J . B . Remmel , *Statistics on wreath products perfect matchings and s igned words* , European Journal of Combinatorics , to appear .

[9] P . A . MacMahon , *Combinatory analysis* , vol . 1 , Cambridge Univ . Press , Cambridge , 1 9 1 8 .

[1 0] P . A . MacMahon , *The indices of permutations and the derivation therefrom of functions of a s ingle varaible ass coiated with the permutations of any ass emblage of o bjects* , Amer . J . Math . , 35 (1 9 1 3) , 281 - 322 .

[1 1] P . A . MacMahon , *Two applications of general thorems in combinatory analysis* , Proc . London Math . Soc . , 1 5 (1 9 1 6) , 3 1 4 - 32 1 .

[1 2] V . Reiner , *Upper b inomial posets and s ign ed permutation s tatistics* , Europ . J . Combinatorics , 1 4 , (1 993) , 58 1 – 588 .

[1 3] E . Steingr í msson , *Permutation s tatistics of indexed permutations* , Europ . J . Combin . 1 5 (1 994) , 1 87 – 205 .